Computer Architecture

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Chapter 4: Arithmetic for Computers

[with materials from *COD, RISC-V 2nd Edition*, Patterson & Hennessy 2021,
and M.J. Irwin's presentation, PSU 2008,
The RISC-V Instruction Set Manual, Volume I, ver. 2.2]

What are stored inside computer?

- Data, of course!
  - Audio, video, image, drawings,…
  - Documents, personal information,…
  - Finance record, corporate business data,…
  - …
- Complex data is constructed from basic data types.
  - Integers
  - Real numbers (Floating point)
  - Symbols (Characters)
- All are represented as binary numbers.

Content

- (Super) Basics of logic design
- Integer representation
- Integer arithmetic (inside computer)
- Floating point number representation and arithmetic

Unsigned Binary Integers

- Using n-bit binary number to represent non-negative integer

$$x = x_{n-1}x_{n-2}\ldots x_1x_0$$
$$= x_{n-1}2^{n-1} + x_{n-2}2^{n-2} + \cdots + x_1 2^1 + x_0 2^0$$

- **Range: 0 to $+2^n - 1$**

- **Example**

$0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 1011_2$
$= 0 + \ldots + 1\times2^3 + 0\times2^2 + 1\times2^1 + 1\times2^0$
$= 0 + \ldots + 8 + 0 + 2 + 1 = 11_{10}$

- **Data range using 32 bits**

0 to 2^{32}-1 = 4,294,967,295

Eg: 32 bit Unsigned Binary Integers

Hex	Binary	Decimal
0x00000000	0…0000	0
0x00000001	0…0001	1
0x00000002	0…0010	2
0x00000003	0…0011	3
0x00000004	0…0100	4
0x00000005	0…0101	5
0x00000006	0…0110	6
0x00000007	0…0111	7
0x00000008	0…1000	8
0x00000009	0…1001	9
	…	
0xFFFFFFFC	1…1100	2^{32}-4
0xFFFFFFFD	1…1101	2^{32}-3
0xFFFFFFFE	1…1110	2^{32}-2
0xFFFFFFFF	1…1111	2^{32}-1

Exercise

- Convert from decimal to 32-bit binary integers

  25 = 0000 0000 0000 0000 0000 0000 0001 1001

  125 = 0000 0000 0000 0000 0000 0000 0111 1101

  255 = 0000 0000 0000 0000 0000 0000 1111 1111

- Convert 32-bit binary integers to decimal

  0000 0000 0000 0000 0000 0000 1100 1111 = 207

  0000 0000 0000 0000 0000 0001 0011 0011 = 307

Signed binary integers

- Using n-bit binary number to represent integer, including negative values

$$x = x_{n-1}x_{n-2}\ldots x_1x_0$$
$$= -x_{n-1}2^{n-1} + x_{n-2}2^{n-2} + \cdots + x_1 2^1 + x_0 2^0$$

- **Range: -2^{n-1} to $+2^{n-1} - 1$**
- **The left most bit (msb) indicates the sign of the number**
- **Example**

  $1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1100_2$
  $= -1\times2^{31} + 1\times2^{30} + \ldots + 1\times2^2 + 0\times2^1 + 0\times2^0$
  $= -2{,}147{,}483{,}648 + 2{,}147{,}483{,}644 = -4_{10}$

- **Using 32 bits**

  $-2{,}147{,}483{,}648$ to $+2{,}147{,}483{,}647$

Signed integer negation

- Given $x = x_{n-1}x_{n-2}\ldots x_1x_0$, how to calculate $-x$?
- Let $\bar{x} = 1's\ complement\ of\ x$

$$\bar{x} = 1111\ldots11_2 - x$$

(1 → 0, 0 → 1)

Then

$$\bar{x} + x = 1111\ldots112 = -1$$

➔ $$\bar{x} + 1 = -x$$

- The computer uses 2's complement form to represent a negative number

- Example: find binary representation of -2

  +2 = 0000 0000 … 0010_2

  –2 = 1111 1111 … 1101_2 + 1

  = 1111 1111 … 1110_2

Exercise

- Find 16 bit signed integer representation of

16 = 0000 0000 0001 0000

-16 = 1111 1111 1111 0000

100 = 0000 0000 0110 0100

-100 = 1111 1111 1001 1100

Sign extension

- Given n-bit integer $x = x_{n-1}x_{n-2}\ldots x_1x_0$
- Find corresponding m-bit representation (m > n) with the same numeric value

$$x = x_{m-1}x_{m-2}\ldots x_1x_0$$

- → Replicate the sign bit to the left

- Examples: 8-bit to 16-bit

  +2: 0000 0010 => 0000 0000 0000 0010

  –2: 1111 1110 => 1111 1111 1111 1110

Instruction to work with sign/unsigned

- lb/lbu, lh/lhu
- blt/bltu, bge/bgeu
- slt/sltu, slti/sltiu
- div/divu, rem/remu

Example

- What is the output of the following program?
- What if the blt instruction is replaced by bltu?

```
        li t0, 20
        li t1, -20
        bltu t1, t0, else
        li a0, 1
        j print
else:
        li a0, 0
print:
        li a7, 1
        ecall
```

Example

- What are the decimal values of s0, s1, s2, s3?

```
.data
        x: .byte 20
        y: .byte -20
.text
        la t0, x
        la t1, y
        lb  s0, 0(t0)
        lbu s1, 0(t0)
        lb  s2, 0(t1)
        lbu s3, 0(t1)
```

Addition and subtraction

- Addition
  - Similar to what you do to add two numbers manually
  - Digits are added bit by bit from right to left
  - Carries passed to the next digit to the left
- Subtraction
  - Negate the second operand then add to the first operand

$$
\begin{array}{rl}
 & 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0111_{two} = 7_{ten} \\
+ & 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0110_{two} = 6_{ten} \\
\hline
 & 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 1101_{two} = 13_{ten}
\end{array}
$$

Carryout and Overflow

- Carryout: adding or substracting n-bit binary numbers result in carryout to or borrow from bit n+1.
- Overflow: adding or subtracting n-bit signed integers result in a value that cannot be represented by a n-bit signed integer.
  - When adding operands with different signs or when subtracting operands with the same sign, overflow can *never* occur

Operation	Operand A	Operand B	Result indicating overflow
A + B	≥ 0	≥ 0	< 0
A + B	< 0	< 0	≥ 0
A - B	≥ 0	< 0	< 0
A - B	< 0	≥ 0	≥ 0

Overflow consequences

- Incorrect results: the result may
  - wrap around (unsigned numbers), or
  - change sign (signed numbers)
- Hardware exception:
  - In Intel x86: the overflow flag (a specific bit in a global flag register) is set when overflow occurs, allowing software to detect it.
  - In RISC-V: hardware does not care about the overflow → must be handled explicitly by software

Examples

- All numbers are 8-bit signed integer

```
 12 +  8 =
122 +  8 =
122 + 80 =
122 - 80 =
```

Arithmetic from hardware viewpoint

- How does computer implement the addition by hardware?
  - Answer: the computer implements logic circuits
  - Logic circuits are built using logic gates: AND, OR, NOT
  - Supported theory: Boolean algebra

Basics of logic design (Appendix A)

- Boolean logic: logic variable and operators
- Logic variable: values of 1 (TRUE) or 0 (FALSE)
- Basic operators: AND, OR, NOT
  - A AND B : $A \cdot B$ hay AB
  - A OR B : $A + B$
  - NOT A : $\overline{A}$
  - Order: NOT > AND > OR
- Additional operators: NAND, NOR, XOR
  - A NAND B: $\overline{A \cdot B}$
  - A NOR B : $\overline{A + B}$
  - A XOR B: $A \oplus B = A \bullet \overline{B} + \overline{A} \bullet B$

Truth tables

A	B	A AND B A•B
0	0	0
0	1	0
1	0	0
1	1	1

A	B	A OR B A + B
0	0	0
0	1	1
1	0	1
1	1	1

A	NOT A $\bar{A}$
0	1
1	0

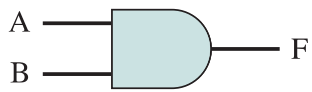
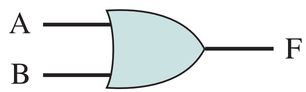
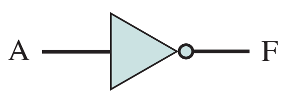
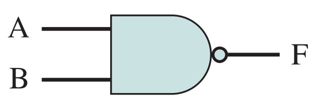
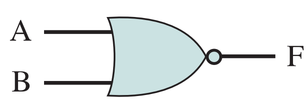
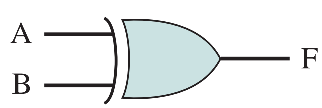
Unary operator NOT

A	B	A NAND B $\overline{A.B}$
0	0	1
0	1	1
1	0	1
1	1	0

A	B	A XOR B A ⊕ B
0	0	0
0	1	1
1	0	1
1	1	0

A	B	A NOR B $\overline{A+B}$
0	0	1
0	1	0
1	0	0
1	1	0

Logic gates

Name	Graphical Symbol	Algebraic Function	Truth Table
AND		$F = A \bullet B$ or $F = AB$	A B | F 0 0 | 0 0 1 | 0 1 0 | 0 1 1 | 1
OR		$F = A + B$	A B | F 0 0 | 0 0 1 | 1 1 0 | 1 1 1 | 1
NOT		$F = \overline{A}$ or $F = A'$	A | F 0 | 1 1 | 0
NAND		$F = \overline{AB}$	A B | F 0 0 | 1 0 1 | 1 1 0 | 1 1 1 | 0
NOR		$F = \overline{A + B}$	A B | F 0 0 | 1 0 1 | 0 1 0 | 0 1 1 | 0
XOR		$F = A \oplus B$	A B | F 0 0 | 0 0 1 | 1 1 0 | 1 1 1 | 0

Laws of Boolean algebra

$A \bullet B = B \bullet A$	$A + B = B + A$
$A \bullet (B + C) = (A \bullet B) + (A \bullet C)$	$A + (B \bullet C) = (A + B) \bullet (A + C)$
$1 \bullet A = A$	$0 + A = A$
$A \bullet \overline{A} = 0$	$A + \overline{A} = 1$
$0 \bullet A = 0$	$1 + A = 1$
$A \bullet A = A$	$A + A = A$
$A \bullet (B \bullet C) = (A \bullet B) \bullet C$	$A + (B + C) = (A + B) + C$
$\overline{A \bullet B} = \overline{A} + \overline{B}$ (DeMorgan's law)	$\overline{A + B} = \overline{A} \bullet \overline{B}$ (DeMorgan's law)

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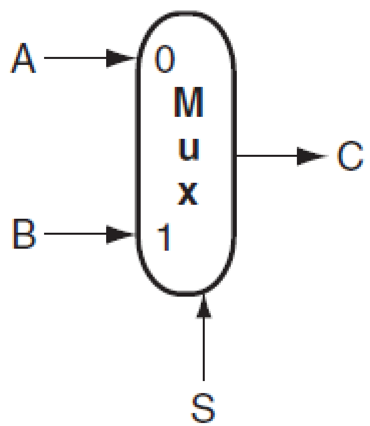
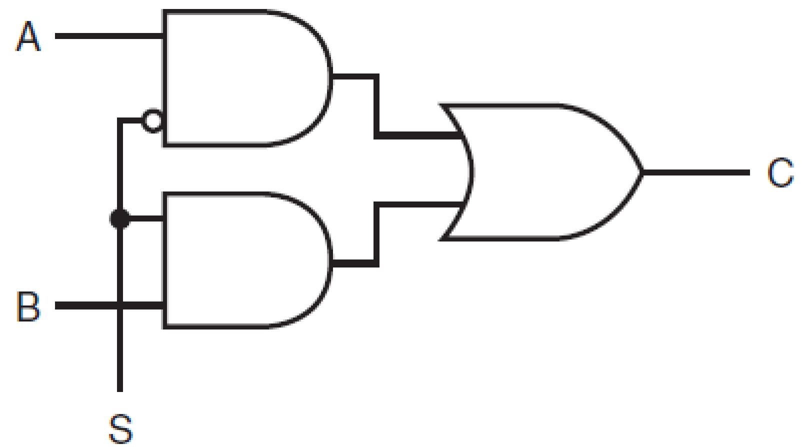
Example: multiplexor

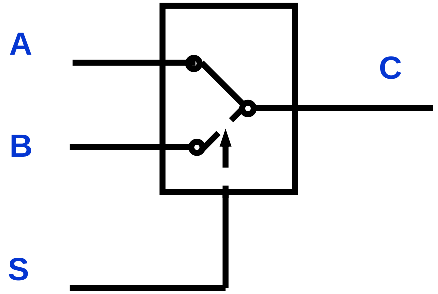
- Depending on S, output C is equal to one of the two inputs A, B
- Explain how this circuit works?





Multiplexor explaination

MUX 2-1



S	C
0	A
1	B

S \ AB	00	01	11	10
0		1	1	
1			1	1

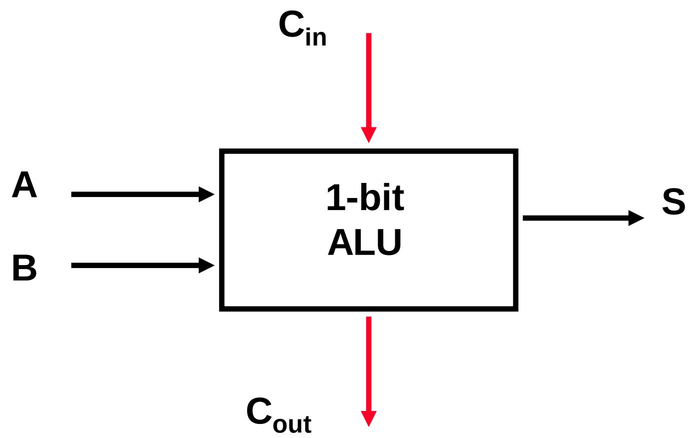
Truth table

S	B	A	C
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

$$C = A\bar{S} + \mathrm{BS}$$

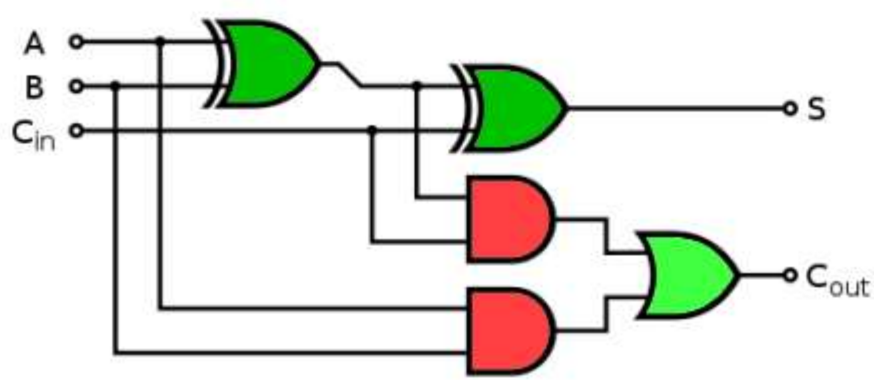
Adder implementation

- 1-bit full adder



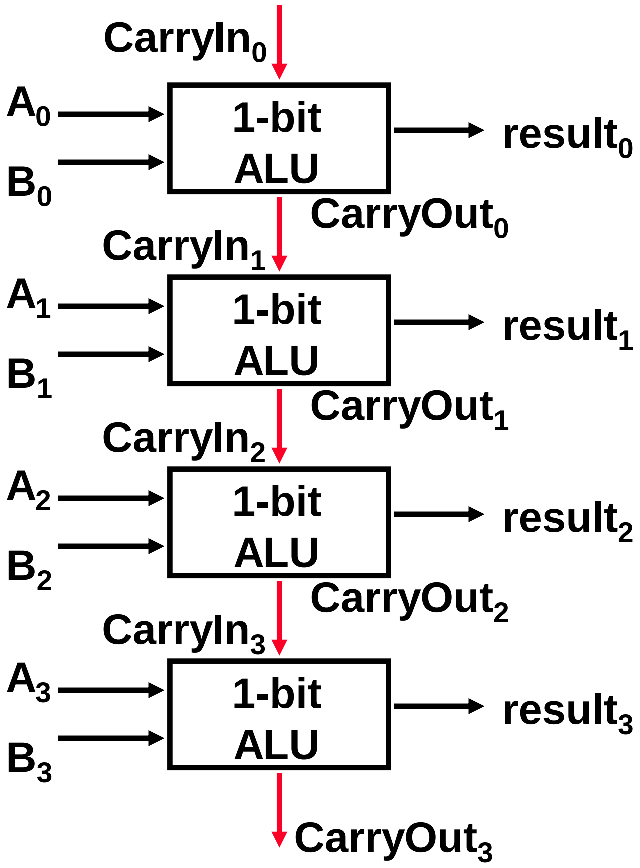
Inputs			Outputs	
A	B	Cin	S	Cout
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

- $S = Cin \oplus (A \oplus B)$
- $C_{out} = AB + BCin + ACin$



Adder implementation

- N-bit ripple-carry adder



Performance depends on data length

➔ Performance is low

Making addition faster: infinite hardware

- Parallelize the adder with the cost of hardware
- Given the addition:

$$a_{n-1}a_{n-2}\ldots a_1a_0 + bn_{-1}b_{n-2}\ldots b_1b_0$$

- Let c_i is the carry at bit i

$$\mathrm{c2} = (\mathrm{b1}.\mathrm{c1}) + (\mathrm{a1}.\mathrm{c1}) + (\mathrm{a1}.\mathrm{b1})$$
$$\mathrm{c1} = (\mathrm{b0}.\mathrm{C0}) + (\mathrm{a0}.\mathrm{c0}) + (\mathrm{a0}.\mathrm{b0})$$

Find c2 from a0, b0, a1, b1?

Making addition faster: Carry Lookahead

- Video demo: https://www.youtube.com/watch?v=yj6wo5SCObY
- Approach
  - Make hardwired 4 bit adder → fast and simple enough
  - Develop a carry lookahead unit to calculate the carry bit before finishing the addition
- At bit *i*

$$c_{i+1} = (b_i \cdot c_i) + (a_i \cdot c_i) + (a_i \cdot b_i)$$
$$= (a_i \cdot b_i) + (a_i + b_i) \cdot c_i$$

- Denote

$$g_i = a_i \cdot b_i$$
$$p_i = a_i + b_i$$

- Then

$$c_{i+1} = g_i + p_i \cdot c_i$$

Carry lookahead

- With 4-bit adder

$$c1 = g0 + (p0 \cdot c0)$$

$$c2 = g1 + (p1 \cdot g0) + (p1 \cdot p0 \cdot c0)$$

$$c3 = g2 + (p2 \cdot g1) + (p2 \cdot p1 \cdot g0) + (p2 \cdot p1 \cdot p0 \cdot c0)$$

$$c4 = g3 + (p3 \cdot g2) + (p3 \cdot p2 \cdot g1) + (p3 \cdot p2 \cdot p1 \cdot g0) + (p3 \cdot p2 \cdot p1 \cdot p0 \cdot c0)$$

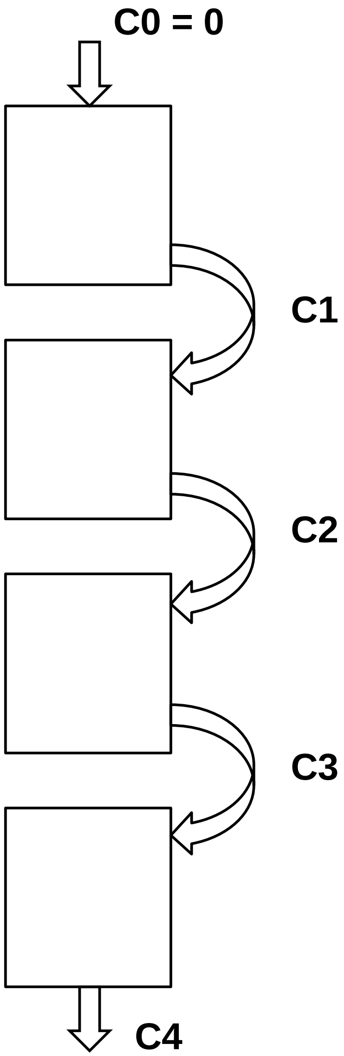
→ All carry bits can be calculated after 3 gate delay

→ All result bits can be calculated after maximum of 4 gate delay

→ How to implement bigger adder?

Carry lookahead

- For 16-bit adder → fast C1, C2, C3, C4 is needed



Carry lookahead

- Denote

$$P0 = p3 \cdot p2 \cdot p1 \cdot p0 \qquad G0 = g3 + (p3 \cdot g2) + (p3 \cdot p2 \cdot g1) + (p3 \cdot p2 \cdot p1 \cdot g0)$$

$$P1 = p7 \cdot p6 \cdot p5 \cdot p4 \qquad G1 = g7 + (p7 \cdot g6) + (p7 \cdot p6 \cdot g5) + (p7 \cdot p6 \cdot p5 \cdot g4)$$

$$P2 = p11 \cdot p10 \cdot p9 \cdot p8 \qquad G2 = g11 + (p11 \cdot g10) + (p11 \cdot p10 \cdot g9) + (p11 \cdot p10 \cdot p9 \cdot g8)$$

$$P3 = p15 \cdot p14 \cdot p13 \cdot p12 \qquad G3 = g15 + (p15 \cdot g14) + (p15 \cdot p14 \cdot g13) + (p15 \cdot p14 \cdot p13 \cdot g12)$$

- Then big-carry bits can be calculated fast

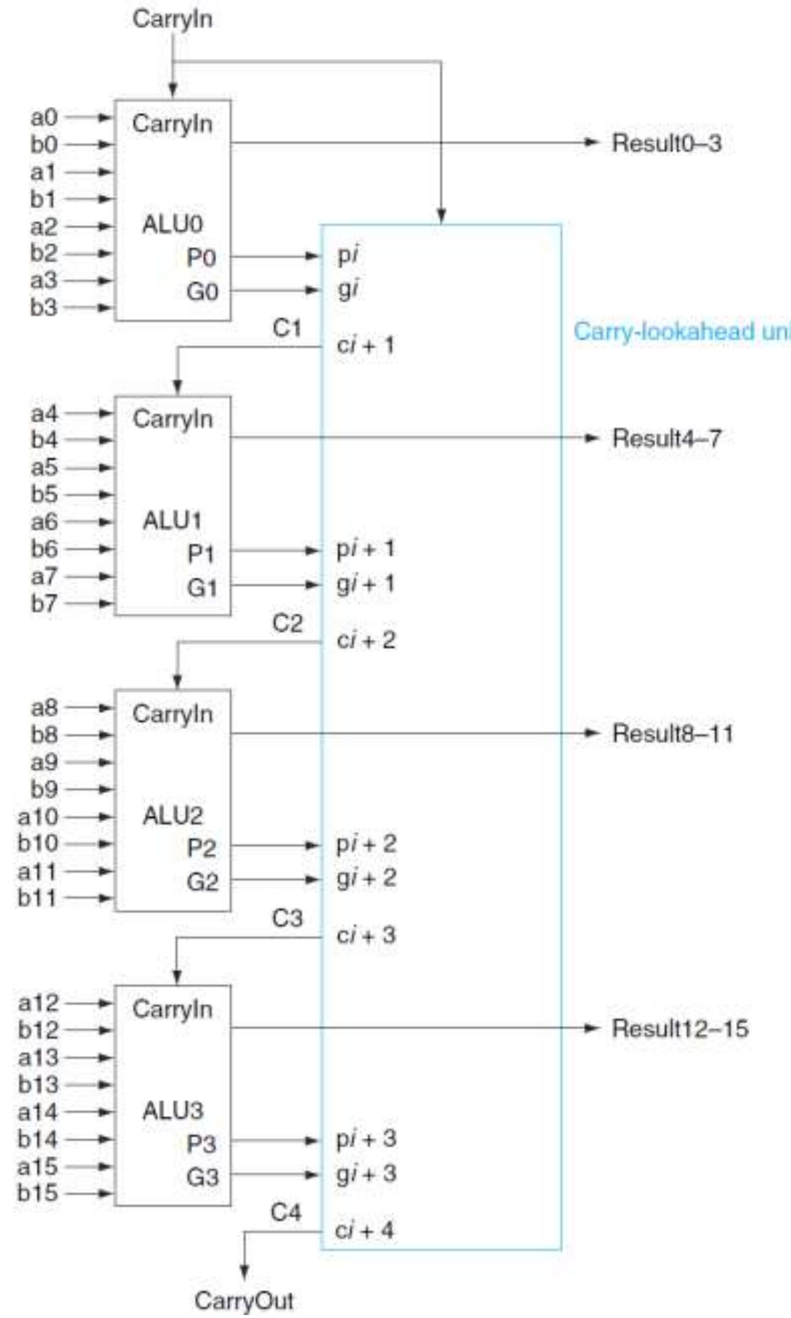
$$C1 = G0 + (P0 \cdot c0)$$

$$C2 = G1 + (P1 \cdot G0) + (P1 \cdot P0 \cdot c0)$$

$$C3 = G2 + (P2 \cdot G1) + (P2 \cdot P1 \cdot G0) + (P2 \cdot P1 \cdot P0 \cdot c0)$$

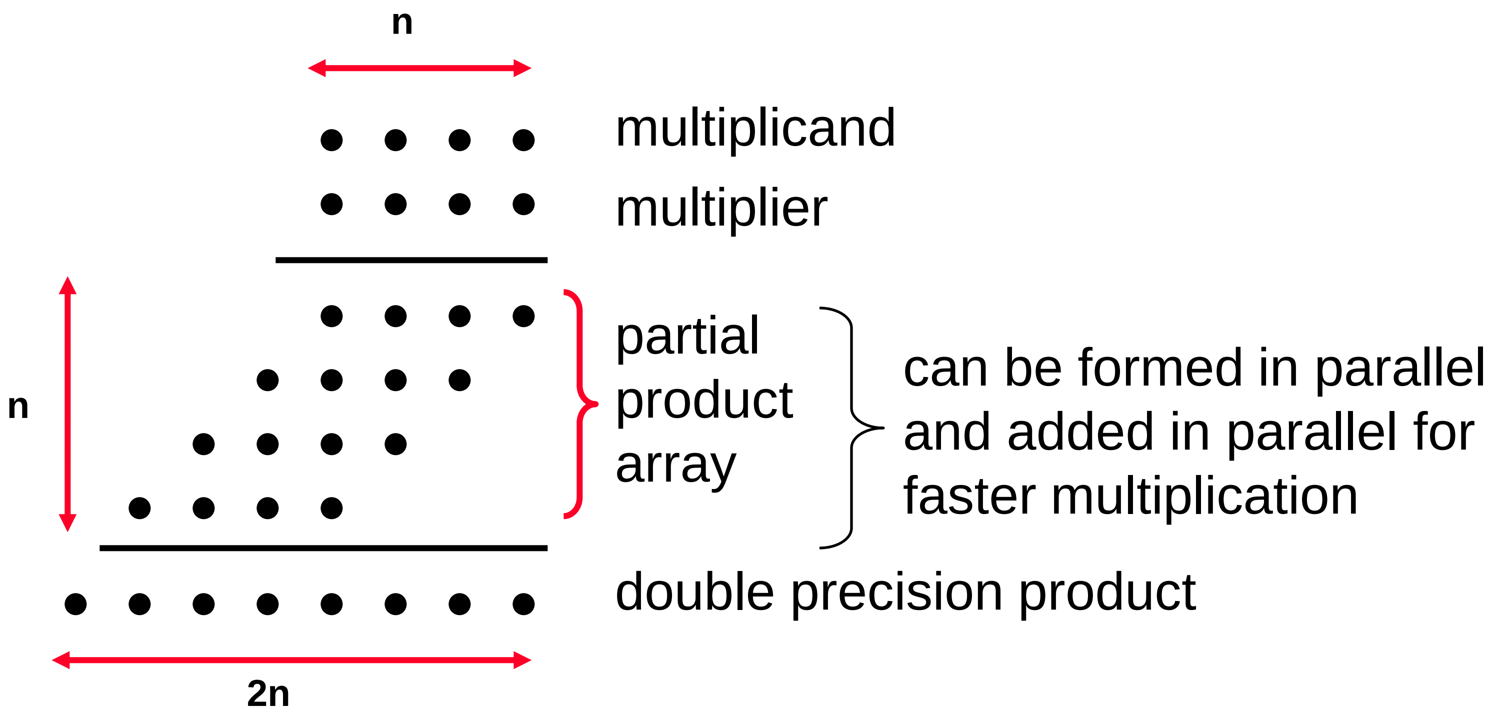
$$C4 = G3 + (P3 \cdot G2) + (P3 \cdot P2 \cdot G1) + (P3 \cdot P2 \cdot P1 \cdot G0) + (P3 \cdot P2 \cdot P1 \cdot P0 \cdot c0)$$

16-bit Adder


CarryIn
a0
b0
a1
b1
a2
b2
a3
b3
CarryIn
ALU0
P0
G0
Result0–3
pi
gi
C1
ci + 1
Carry-lookahead unit
a4
b4
a5
b5
a6
b6
a7
b7
CarryIn
ALU1
P1
G1
Result4–7
pi + 1
gi + 1
C2
ci + 2
a8
b8
a9
b9
a10
b10
a11
b11
CarryIn
ALU2
P2
G2
Result8–11
pi + 2
gi + 2
C3
ci + 3
a12
b12
a13
b13
a14
b14
a15
b15
CarryIn
ALU3
P3
G3
Result12–15
pi + 3
gi + 3
C4
ci + 4
CarryOut

Multiply

- Binary multiplication is just a *bunch* of right shifts and adds



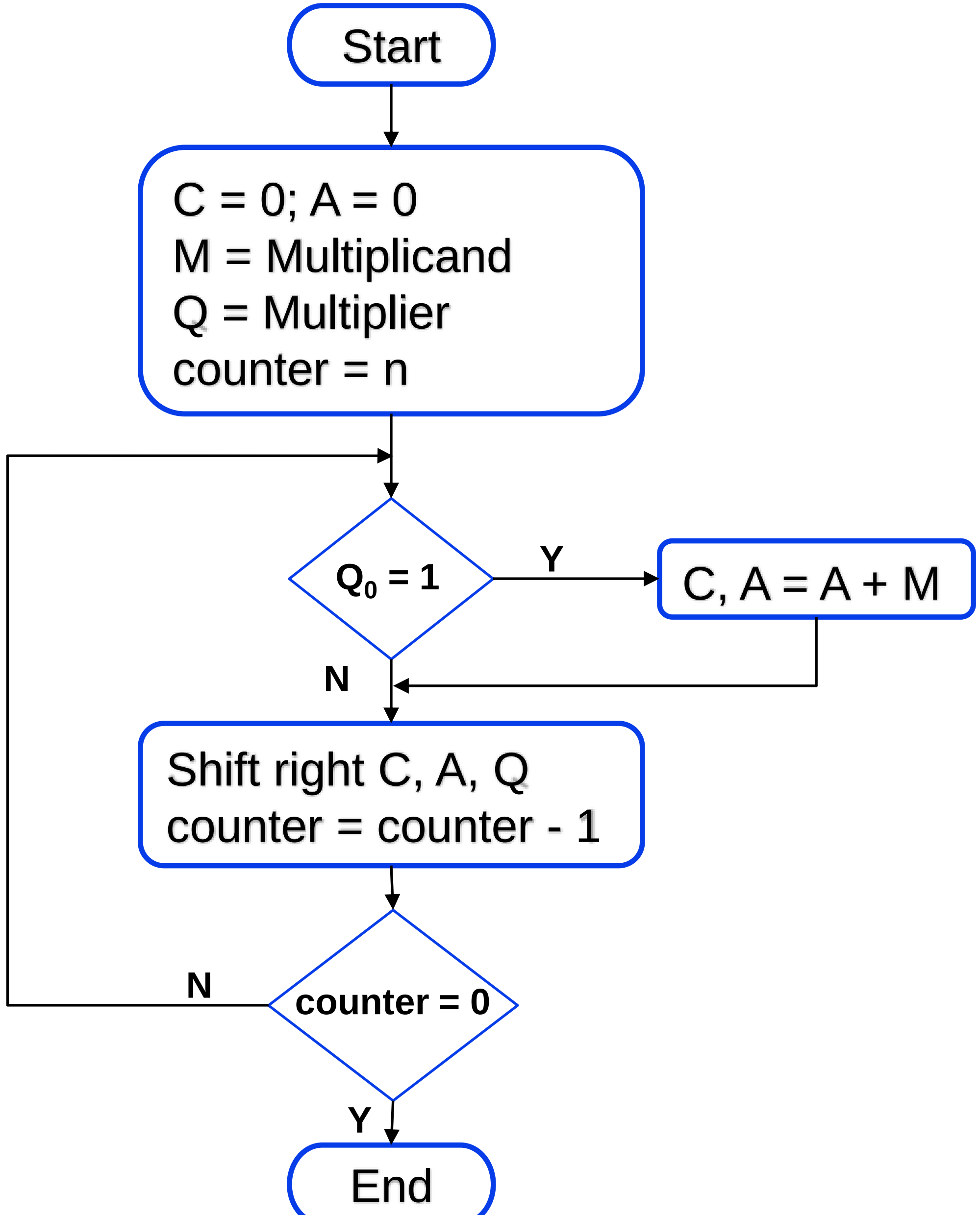
n-bit multiplicand and multiplier → 2n-bit product

Example

Multiplicand		1000_{ten}
Multiplier	x	1001_{ten}
		1000
		0000
		0000
		1000
Product		1001000_{ten}

How to do this in hardware?

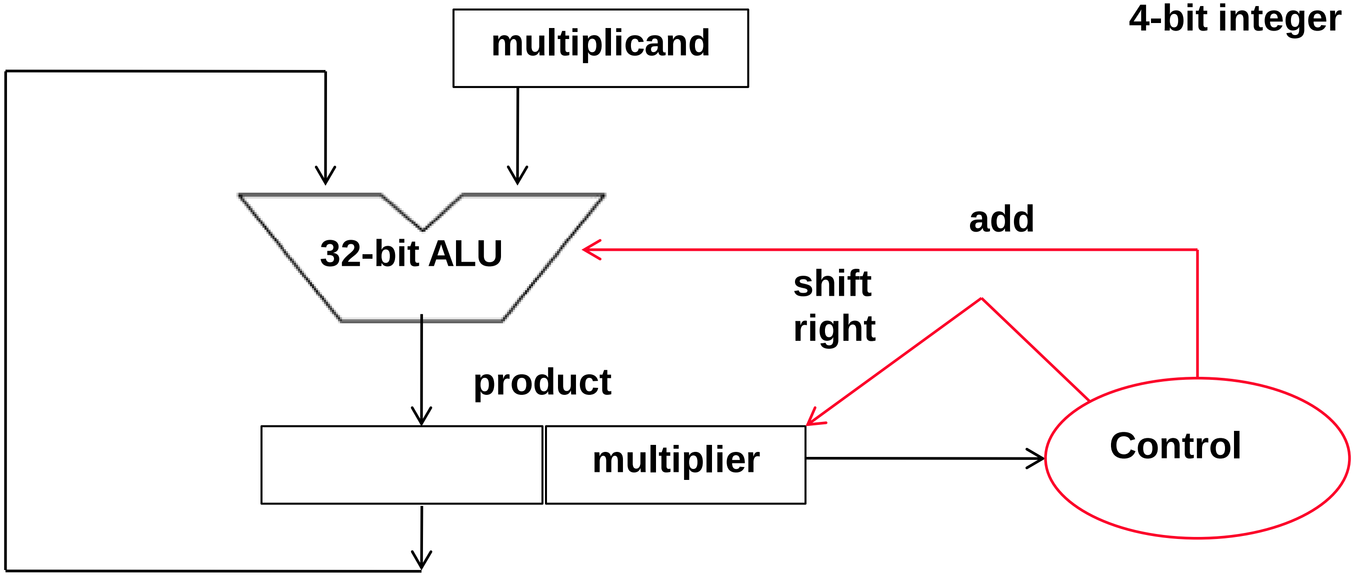
Add and Right Shift Multiplier



Add and Right Shift Multiplier Hardware

0 1 1 0 = 6

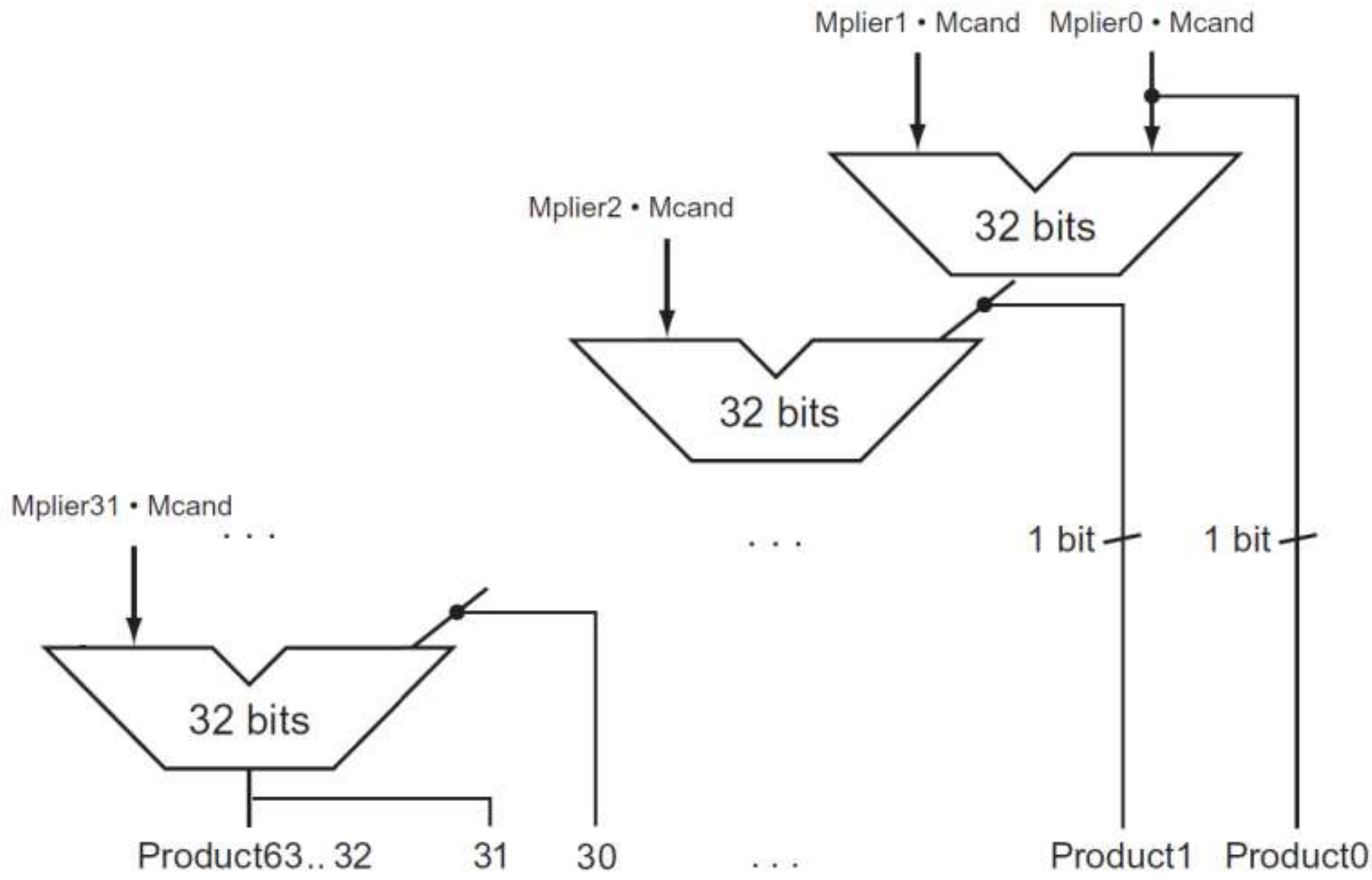
6 x 5 = ?
4-bit integer



	0 0 0 0	0 1 0 1	= 5	
add	0 1 1 0	0 1 0 1	LSB=1 → add multiplicand	
	0 0 1 1 →	0 0 1 0	shift right	
add	0 0 1 1	0 0 1 0	LSB=0 → no change	
	0 0 0 1 →	1 0 0 1	shift right	
add	0 1 1 1	1 0 0 1	LSB=1 → add multiplicand	
	0 0 1 1 →	1 1 0 0	shift right	
add	0 0 1 1	1 1 0 0	LSB=0 → no change	
	0 0 0 1 →	1 1 1 0	shift right	= 30

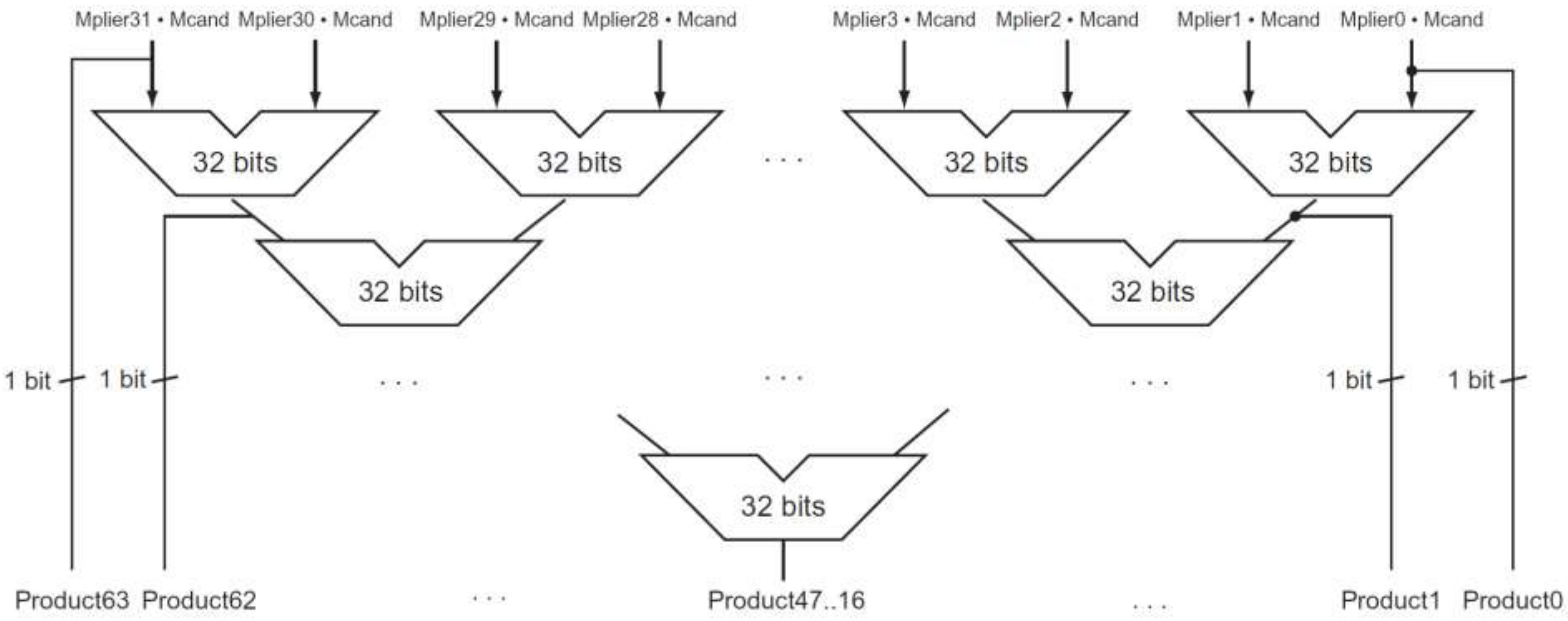
Fast multiplier – Design for Moore

- Why is this fast?



Fast multiplier – Design for Moore

- How fast is this?
- Note: the size of addition circuits



RISC-V Multiply Instruction (RV32M extension)

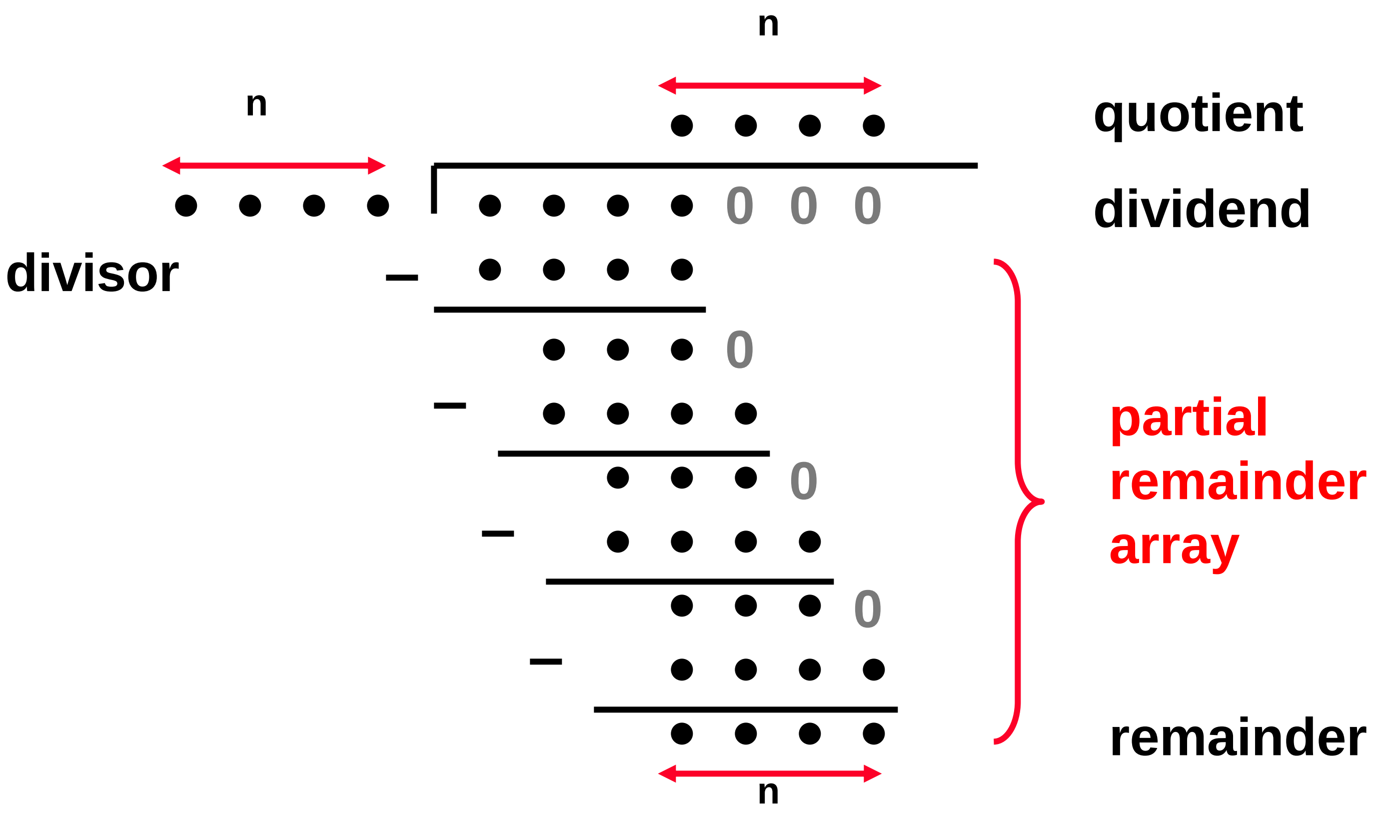
- Multiply instructions: `mul, mulh mulhu, mulhsu`

```
mul  t1, s0, s1 #set t1 to lower 32 bits
                 of s0 * s1

mulh t1, s0, s1 #set t1 to upper 32 bits
                 of s0 * s1

mulhu t1, s0, s1 #set t1 to upper 32 bits of
            s0 * s1(unsigned multiplication)

mulhsu t1, s0, s1 #set t1 to upper 32 bits
  of s0 * s1, where s0 is signed and s1 is
  unsigned
```

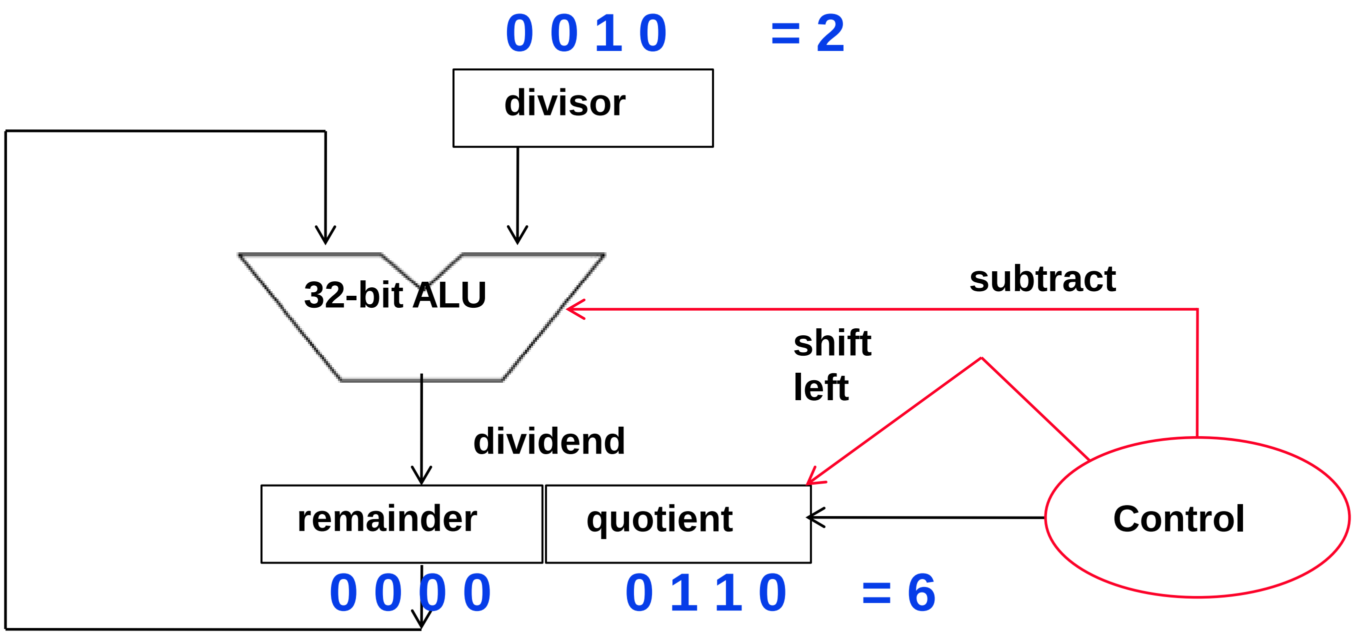
Division

- Division is just a *bunch* of quotient digit guesses and left shifts and subtracts

dividend = quotient x divisor + remainder



Left Shift and Subtract Division Hardware



	0 0 0 0	1 1 0 0	
sub	1 1 1 0	1 1 0 0	rem<0, so quotient bit = 0
	0 0 0 0	1 1 0 0	and restore remainder
	0 0 0 1	1 0 0 0	
sub	1 1 1 1	1 0 0 0	rem<0, so quotient bit = 0
	0 0 0 1	1 0 0 0	and restore remainder
	0 0 1 1	0 0 0 0	
sub	0 0 0 1	0 0 0 1	rem≥0, so quotient bit = 1
	0 0 1 0	0 0 1 0	
sub	0 0 0 0	0 0 1 1	rem≥0, so quotient bit = 1
			all bits of divident are processed

Result: remainder 0, quotient 3

RISC-V Divide Instruction (RV32M extension)

- Instructions:

```
div   t1, t2, t3 #t1 = t2/t3 (signed division)

divu t1, t2, t3 #t1 = t2/t3 (unsigned division)

rem  t1, t2, t3 #t1 = remainder of t2/t3

remu t1, t2, t3 #t1 = remainder of t2/t3
              (unsigned)
```

- As with multiply, divide ignores overflow so software must determine if the quotient is too large.
- Software must also check the divisor to avoid division by 0.

Signed integer multiplication and division

- Reuse unsigned multiplication then fix product sign later
- Multiplication
  - Multiplicand and multiplier are of the same sign: keep product
  - Multiplicand and multiplier are of different sign: negate product
- Division:
  - Dividend and divisor of the same sign:
    - Keep quotient
    - Keep/negate remainder so it is of the same sign with dividend
  - Dividend and divisor of different sign:
    - Negate quotient
    - Keep/negate remainder so it is of the same sign with dividend

Example

- Write a RISC-V program
  - Reads 2 integers a and b from console
  - Print out the two values: (a / b) and (a % b) to console

Exercise

- Write a program that
  - Reads two integers a and b from console.
  - Find and print out the greatest common divisor of a and b.
  - Find and print out the least common multiplier of a and b.

Representing Big (and Small) Numbers

- Encoding non-integer value?
  - Earth mass: $(5.9722 \pm 0.0006) \times 10^{24}$ (kg)
  - Weight of an amu (atomic mass unit, 1/12 mass of C_{12})

    0.000000000000000000000000000166 or 1.6×10^{-27} (kg)
  - PI number

    PI = 3.14159….
- Problem: how to represent the above numbers?

→ We need reals or floating-point numbers!

→ Floating point numbers in decimal:
  - → 1000
  - → 1×10^3
  - → 0.1×10^4

Floating point number

- In decimal system

$$2013.1228 = 201.31228 * 10$$
$$= 20.131228 * 10^2$$
$$= 2.0131228 * 10^3$$
$$= 20131228 * 10^{-4}$$

- What is the “standard” form?

$$2.0131228 * 10^3 = \underbrace{2.0131228}_{\text{mantissa}}\text{E+}\underbrace{03}_{\text{exponent}}$$

- In binary $X = \pm 1.xxxxx * 2^{yyyy}$
- ***Sign, mantissa, and exponent need to be represented***

Floating point number

- Defined by the IEEE 754-1985 standard
  - Single precision: 32 bit
  - Double precision: 64 bit
  - Correspond to float and double in C
- Single precision floating point representation

$$(-1)^{sign} \times 1.F \times 2^{E-bias}$$

  - Fit everything in 32 bits
  - Bias = 127 (with single precision)

s	E (exponent)	F (fraction)
1 sign bit	**8 bits**	**23 bits**

Examples

- Ex1: convert X into decimal value

X = 1100 0001 0101 0110 0000 0000 0000 0000

sign = 1 → X is negative

E = 1000 0010 = 130

F = 10101100...00

→ X = $(-1)^1$ x 1.101011000..00 x $2^{130-127}$

= -1.101011 x 2^3 = -1101.011

= -13.375

Example

- Ex2: find decimal value of X

X = 0011 1111 1000 0000 0000 0000 0000 0000

sign = 0

E = 0111 1111 = 127

F = 000…0000 (23 bit 0)

$X = (-1)^0 \times 1.00…000 \times 2^{127-127} = 1.0$

Example

- Ex3: find binary representation of X = 9.6875 in IEEE 754 single precision

Converting X to plain binary

$$9_{10} = 1001_2$$

$0.6875 \times 2 = 1.375$ → get bit 1
$0.375 \times 2 = 0.75$ → get bit 0
$0.75 \times 2 = 1.5$ → get bit 1
$0.5 \times \quad = 1.0$ → get bit 1

→ $9.6875_{10} = 1001.1011_2$

Example

- Ex3: find binary representation of X = 9.6875 in IEEE 754 single precision

$X = 9.6875_{(10)} = 1001.1011_{(2)} = 1.0011011 \times 2^3$

Then

$S = 0$

$E = 127 + 3 = 130_{(10)} = 1000\ 0010_{(2)}$

F = 001101100...00 (23 bit)

Finally

X = 0100 0001 0001 1011 0000 0000 0000 0000

Examples

- $1.0_2 \times 2^{-1} =$
- $100.75_{10} =$

Some special values

- Smallest+: 0 00000001 1.00000000000000000000000

  $= 1 \times 2^{1-127}$

- Zero: 0 00000000 00000000000000000000000

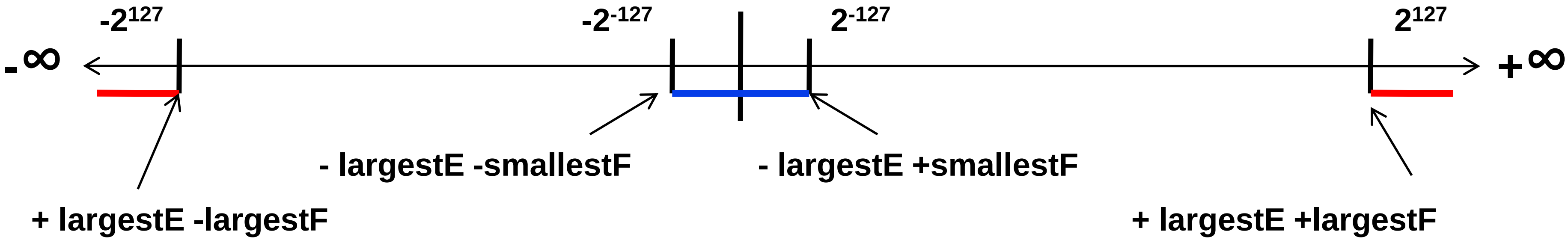
  = true 0

- Largest+: 0 11111110 1.11111111111111111111111

  $= (2-2^{-23}) \times 2^{254-127}$

Too large or too small values

- Overflow (floating point) happens when a positive exponent becomes too large to fit in the exponent field
- Underflow (floating point) happens when a negative exponent becomes too large to fit in the exponent field



- **Double precision: 64 bits**

s	E (exponent)	F (fraction)
1 bit	11 bits	20 bits

F (fraction continued)
32 bits

Too large or too small values

- Question:
  - How to represent a number less than 2^{-127}?
- Answers: use de-normalized representation
  - All bits of E are zero: 00000000
  - One bit of M is not zero, i.e., F is nonzero
  - Value representation:

$$0.\text{fraction} \times 2^{-126}$$

IEEE 754 FP Standard Encoding

- Special encodings are used to represent unusual events
  - ± infinity for division by zero
  - NAN (not a number) for invalid operations such as 0/0
  - True zero is the bit string all zero

Single Precision		Double Precision		Object Represented
E (8)	F (23)	E (11)	F (52)	
0000 0000	0	0000 … 0000	0	true zero (0)
0000 0000	nonzero	0000 … 0000	nonzero	± denormalized number
0111 1111 to +127,-126	anything	0111 …1111 to +1023,-1022	anything	± floating point number
1111 1111	+ 0	1111 … 1111	- 0	± infinity
1111 1111	nonzero	1111 … 1111	nonzero	not a number (NaN)

Floating Point Addition

- Addition (and subtraction)

$$(\pm F1 \times 2^{E1}) + (\pm F2 \times 2^{E2}) = \pm F3 \times 2^{E3}$$

  - **Step 0: Restore the hidden bit in F1 and in F2**
  - **Step 1: Align fractions by right shifting F2 by E1 - E2 positions (assuming E1 $\geq$ E2) keeping track of (three of) the bits shifted out in G R and S**
  - **Step 2: Add the resulting F2 to F1 to form F3**
  - **Step 3: Normalize F3 (so it is in the form 1.XXXXX …)**
    - **If F1 and F2 have the same sign $\rightarrow$ F3 $\in[1,4)$ $\rightarrow$ 1 bit right shift F3 and increment E3 (check for overflow)**
    - **If F1 and F2 have different signs $\rightarrow$ F3 may require *many* left shifts each time decrementing E3 (check for underflow)**
  - **Step 4: Round F3 and possibly normalize F3 again**
  - **Step 5: Rehide the most significant bit of F3 before storing the result**

Floating Point Addition Example

- Add

$$(0.5 = 1.0000 \times 2^{-1}) + (-0.4375 = -1.1100 \times 2^{-2})$$

  - **Step 0:**
  - **Step 1:**
  - **Step 2:**
  - **Step 3:**
  - **Step 4:**
  - **Step 5:**

Floating Point Addition Example

- Add: 0.5 + (-0.4375) = ?

  $(0.5 = 1.0000 \times 2^{-1}) + (-0.4375 = -1.1100 \times 2^{-2})$

  - **Step 0:** **Hidden bits restored in the representation above**
  - **Step 1:** **Shift significand with the smaller exponent (1.1100) right until its exponent matches the larger exponent (so once)**
  - **Step 2:** **Add significands**

    1.000 + (-0.111) = 1.000 – 0.111 = 0.001

  - **Step 3:** **Normalize the sum, checking for exponent over/underflow**

    $\mathbf{0.001 \times 2^{-1} = 0.010 \times 2^{-2} = .. = 1.000 \times 2^{-4}}$

  - **Step 4:** **The sum is already rounded, so we're done**
  - **Step 5:** **Rehide the hidden bit before storing**

Floating Point Multiplication

- Multiplication

$$(\pm F1 \times 2^{E1}) \text{ x } (\pm F2 \times 2^{E2}) = \pm F3 \times 2^{E3}$$

  - **Step 0: Restore the hidden bit in F1 and in F2**
  - **Step 1: Add the two (biased) exponents and subtract the bias from the sum, so E1 + E2 – 127 = E3**

    also determine the sign of the product (which depends on the sign of the operands (most significant bits))
  - **Step 2: Multiply F1 by F2 to form a double precision F3**
  - **Step 3: Normalize F3 (so it is in the form 1.XXXXX …)**
    - **Since F1 and F2 come in normalized $\rightarrow$ F3 $\in$[1,4) $\rightarrow$ 1 bit right shift F3 and increment E3**
    - **Check for overflow/underflow**
  - **Step 4: Round F3 and possibly normalize F3 again**
  - **Step 5: Rehide the most significant bit of F3 before storing the result**

Floating Point Multiplication Example

- Multiply

$$(0.5 = 1.0000 \times 2^{-1}) \text{ x } (-0.4375 = -1.1100 \times 2^{-2})$$

  - **Step 0:**
  - **Step 1:**
  - **Step 2:**
  - **Step 3:**
  - **Step 4:**
  - **Step 5:**

Floating Point Multiplication Example

- Multiply

$$(0.5 = 1.0000 \times 2^{-1}) \text{ x } (-0.4375 = -1.1100 \times 2^{-2})$$

- **Step 0:** **Hidden bits restored in the representation above**
- **Step 1:** **Add the exponents (not in bias would be -1 + (-2) = -3 and in bias would be (-1+127) + (-2+127) – 127 = (-1 -2) + (127+127-127) = -3 + 127 = 124**
- **Step 2:** **Multiply the significands**

  1.000 x 1.110 = 1.110000
- **Step 3:** **Normalized the product, checking for exp over/underflow**

  1.110000 x 2^{-3} is already normalized
- **Step 4:** **The product is already rounded, so we're done**
- **Step 5:** **Rehide the hidden bit before storing**

Support for Accurate Arithmetic

- Rounding (except for truncation) requires the hardware to include extra F bits during calculations
  - Guard and Round bit – 2 additional bits to increase accuracy
  - Sticky bit – used to support Round to nearest even; is set to a 1 whenever a 1 bit shifts (right) through it (e.g., when aligning F during addition/subtraction)

F = 1 . xxxxxxxxxxxxxxxxxxxxxxx G R S

- IEEE 754 FP rounding modes
  - Always round up (toward +∞)
  - Always round down (toward -∞)
  - Truncate
  - Round to nearest even (when the Guard || Round || Sticky are 100) – always creates a 0 in the least significant (kept) bit of F

http://pages.cs.wisc.edu/~markhill/cs354/Fall2008/notes/flpt.apprec.html

Example

- Calculate:

  0.2 x 5 = ?

  0.333 x 3 = ?

  (1.0/3) x 3 = ?

Floating point instructions: RV32F

RV32F / D Floating-Point Extensions

Inst	Name	FMT	Opcode	funct3	funct5	Description (C)
flw	Flt Load Word	*				rd = M[rs1 + imm]
fsw	Flt Store Word	*				M[rs1 + imm] = rs2
fmadd.s	Flt Fused Mul-Add	*				rd = rs1 * rs2 + rs3
fmsub.s	Flt Fused Mul-Sub	*				rd = rs1 * rs2 - rs3
fnmadd.s	Flt Neg Fused Mul-Add	*				rd = -rs1 * rs2 + rs3
fnmsub.s	Flt Neg Fused Mul-Sub	*				rd = -rs1 * rs2 - rs3
fadd.s	Flt Add	*				rd = rs1 + rs2
fsub.s	Flt Sub	*				rd = rs1 - rs2
fmul.s	Flt Mul	*				rd = rs1 * rs2
fdiv.s	Flt Div	*				rd = rs1 / rs2
fsqrt.s	Flt Square Root	*				rd = sqrt(rs1)
fsgnj.s	Flt Sign Injection	*				rd = abs(rs1) * sgn(rs2)
fsgnjn.s	Flt Sign Neg Injection	*				rd = abs(rs1) * -sgn(rs2)
fsgnjx.s	Flt Sign Xor Injection	*				rd = rs1 * sgn(rs2)
fmin.s	Flt Minimum	*				rd = min(rs1, rs2)
fmax.s	Flt Maximum	*				rd = max(rs1, rs2)
fcvt.s.w	Flt Conv from Sign Int	*				rd = (float) rs1
fcvt.s.wu	Flt Conv from Uns Int	*				rd = (float) rs1
fcvt.w.s	Flt Convert to Int	*				rd = (int32_t) rs1
fcvt.wu.s	Flt Convert to Int	*				rd = (uint32_t) rs1
fmv.x.w	Move Float to Int	*				rd = *((int*) &rs1)
fmv.w.x	Move Int to Float	*				rd = *((float*) &rs1)
feq.s	Float Equality	*				rd = (rs1 == rs2) ? 1 : 0
flt.s	Float Less Than	*				rd = (rs1 < rs2) ? 1 : 0
fle.s	Float Less / Equal	*				rd = (rs1 <= rs2) ? 1 : 0
fclass.s	Float Classify	*				rd = 0..9

Exercise

- Write the corresponding RISC-V assembly program equivalent to the following C code:

```
float x = 0.75;
printf("%f", x);
```

Solution

```
.data
    x: .float 0.75
.text
    la t1, x
    flw ft0, 0(t1)              #load x from mem

    li a7, 2                 #function 2
    fcvt.s.w ft1, zero       #zero immediate
    fadd.s fa0, ft0, ft1     #move data to fa0
    ecall                    #print
```